# Assignment 4.

This homework is due *Tuesday* Feb 18.

There are total 29 points in this assignment. 26 points is considered 100%. If you go over 26 points, you will get over 100% for this homework (but not over 115%) and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge.* Your solutions should contain full proofs. Bare answers will not earn you much.

Problem 6b is optional.

(1) (a) [1pt] (3.1.5a) Given that $p$ is a prime and $p \mid a^n$, prove that $p^n \mid a^n$.
 (b) [3pt] Let $k > 1$ and $n$ be integers. Prove that $\sqrt[k]{n}$ cannot be rational, unless $n$ is a perfect $k$-th power (i.e. $n = m^k$ for some $m \in \mathbb{Z}$). (*Hint:* If $\frac{a}{b} = \sqrt[k]{n}$, then $a^k = nb^k$. Use (a).)

(2) [3pt] (3.1.5b) If $\gcd(a, b) = p$, a prime, what are possible values of $\gcd(a^2, b^2)$, $\gcd(a^2, b)$ and $\gcd(a^3, b^2)$?

(3) (3.1.3bcd) Prove the following:
 (a) [2pt] Any integer of the form $3n + 2$ has a prime factor of this form.
 (b) [2pt] The only prime of the form $n^3 - 1$ is 7. (*Hint:* Write $n^3 - 1 = (n - 1)(n^2 + n + 1)$.)
 (c) [2pt] The only prime $p$ for which $3p + 1$ is a perfect square is $p = 5$. (*Hint:* Write $3p + 1 = n^2$.)

(4) [3pt] (3.1.8) If $p \geq q \geq 5$ are both primes, prove that $24 \mid p^2 - q^2$. (*Hint:* Show that one of two numbers $p + q, p - q$ is divisible by 4.)

(5) (a) [3pt] Prove that for integer $n > 4$, $n$ is composite if and only if $n \mid (n - 1)!$.
 (b) [2pt] Prove that if $n$ divides $(n - 1)! + 1$, then $n$ is prime. (*Hint:* Use item (a).)

(6) (a) [3pt] Show that if $p_1, p_2, \ldots p_n$ are any distinct primes, then
$$\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_n}$$
is not an integer. (For example: $1/2 + 1/3 + 1/6 = 1 \in \mathbb{Z}$, but 6 is not a prime.)
 (b) [0pt and maybe a cookie] Show that if $n > 1$, then
$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$$
is never an integer.

(7) [2pt] Find all triples of primes of the form $p, p + 2, p + 4$.

(8) [3pt] (3.3.13) Prove that there are infinitely many primes of the form $6n+5$.